I. Introduction

1.3 Find the pressure p both inside and outside the core of the Rankine vortex. Show that the pressure at r = 0 is lower than that at r=∞ by an amount $\rho\Omega^2 a^2$? (hence the very low pressure in the centre of a tornado). Deduce that if there is a free surface to the fluid and gravity is acting, then the surface at r =0 is a depth $\Omega^2 a^2/g$ below the surface at r=∞.

$$\bar{u} = (0, u_\theta, 0)$$

$$u_\theta = \begin{cases} \Omega r, & r \le a \\ \dfrac{\Omega a^2}{r}, & r > a \end{cases}$$

1) Can we use Bernoulli's theorem?

$$\bar{u} = \bar{u}_1 + \underbrace{\bar{u}_2}_{f(\bar{\omega})}$$

$$\underbrace{\bar{\omega} = \bar{\nabla} \times \bar{u}}_{\text{vorticity}}$$

$$\bar{\omega} = 0$$

$r \le a$

$$\bar{u} = u_\theta \bar{e}_\theta = \Omega r(-\sin\theta\, \bar{e}_x + \cos\theta\, \bar{e}_y)$$

$$= -\Omega y \bar{e}_x + \Omega x \bar{e}_y$$

$$\bar{\nabla} \times \bar{u} = 0\bar{e}_x + 0\bar{e}_y + (\Omega + \Omega)\bar{e}_z = 2\Omega \bar{e}_z \ne 0$$

To compute pressure, we just solve Euler eq-s.

$$p(r, z) = \frac{\rho\Omega^2}{2} r^2 - \rho g z + C_1$$

$r > a$

$$\bar{\nabla} \times \bar{u} = \hat{\bar{\nabla}}\times\left(\frac{\Omega a^2}{r}\bar{u}_\theta\right) = \bar{e}_z\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial \cancel{u_r}}{\partial \theta}\right)$$

$$= \bar{e}_z\left(-\frac{\Omega a^2}{r} + \frac{\Omega a^2}{r}\right) = 0$$

$$\frac{p}{\rho} + \frac{1}{2}\bar{u}^2 + gz = \text{const}$$

$$p = -\frac{\rho}{2}\bar{u}^2 - \rho g z + C_2$$

$$= -\frac{\rho \Omega^2 a^4}{2r^2} - \rho g z + C_2$$

2) $p(r=0) = -\rho g z + C_1$

$p(r=\infty) = -\rho g z + C_2$

$p(r=0) - p(r=\infty) = C_1 - C_2$

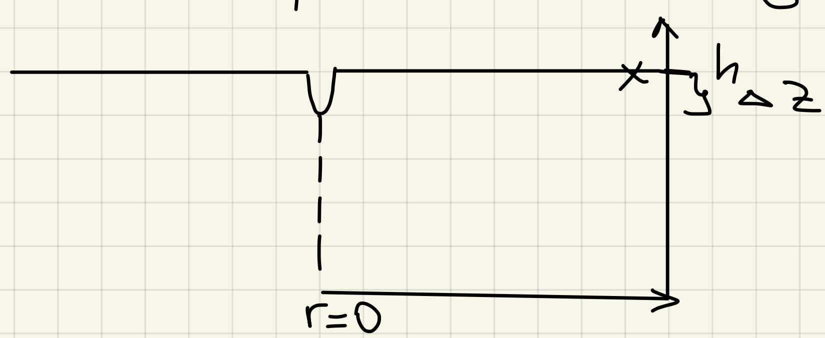
Use the fact that pressure must be continuous

$$\frac{\rho \Omega^2 a^2}{2} - \rho \cancel{g} z + C_1 = -\frac{\rho \Omega^2 a^2}{2} - \rho \cancel{g} z + C_2$$

$$C_1 - C_2 = -\rho \Omega^2 a^2$$

$$p(r=0) - p(r=\infty) = -\rho \Omega^2 a^2$$

3)



$z_s = z(r)$

$$p(0, z) = -\rho g z + C_1 = p_a$$

$$\Rightarrow z_0 = \frac{-p_a + C_1}{\rho g}$$

$$p(\infty, z) = -\rho g z + C_2 = p_a$$

$$\Rightarrow z_1 = \frac{-p_a + C_2}{\rho g}$$

$$\Delta z = z_1 - z_0 = \frac{C_2 - C_1}{\rho g} = \frac{\rho \Omega^2 a^2}{\rho g} = \frac{\Omega^2 a^2}{g}$$

1.4 Take the Euler equation for an incompressible fluid of constant density, cast it into an appropriate form, and perform suitable operations on it to obtain the energy equation:

$$\frac{d}{dt}\int_V \frac{1}{2}\rho \bar{u}^2 dV = -\int_S \left(p' + \frac{1}{2}\rho \bar{u}^2\right)\bar{u}\cdot\bar{n}\, dS,$$

where V is the region enclosed by a fixed closed surface S drawn in the fluid, and p' denotes $p+\rho\chi$, the non-hydrostatic part of the pressure field.

$$\bar{u}\cdot\left[\partial_t \bar{u} + (\bar{u}\cdot\bar{\nabla})\bar{u}\right] = -\bar{u}\cdot\frac{1}{\rho}\bar{\nabla} p$$

1) $\partial_t \bar{u}^2 = 2\bar{u}\cdot\partial_t\bar{u} \Rightarrow \bar{u}\cdot\partial_t\bar{u} = \frac{1}{2}\partial_t \bar{u}^2$

2) $\bar{u}\cdot(\bar{u}\cdot\bar{\nabla})\bar{u} = u_i\, u_j\, \partial_j u_i$

{where $\partial_j = \frac{\partial}{\partial x_j}$}

$$= u_i \partial_j (u_j u_i) = \frac{1}{2}\partial_j (u_j u_i^2) = \frac{1}{2}\bar{\nabla}\cdot(\bar{u}^2\bar{u})$$

$$u_i \partial_j (u_j u_i) = u_i u_j \partial_j u_i + \underbrace{u_i^2 \partial_j u_j}_{\bar{u}^2 \bar{\nabla}\cdot\bar{u}}$$

3) $\bar{u}\cdot\bar{\nabla} p = u_i \partial_i p = \partial_i (u_i p) = \bar{\nabla}\cdot(p\bar{u})$

because $\partial_i u_i = \bar{\nabla}\cdot\bar{u} = 0$

incompr.

$$\frac{\rho}{2}\partial_t \bar{u}^2 + \frac{\rho}{2}\bar{\nabla}\cdot(\bar{u}^2\bar{u}) = -\bar{\nabla}\cdot(p\bar{u})$$

$\rho =$ const

$$\partial_t\left(\frac{\rho}{2}\bar{u}^2\right) + \bar{\nabla}\cdot\left(\frac{\rho}{2}\bar{u}^2\bar{u}\right) + \bar{\nabla}\cdot(p\bar{u}) = 0$$

$$\frac{d}{dt}\int_V \frac{\rho}{2}\bar{u}^2 dV + \underbrace{\int_V \bar{\nabla}\cdot\left(\frac{\rho}{2}\bar{u}^2\bar{u} + p\bar{u}\right)dV}_{\text{divergence th.}} = 0$$

$$\frac{d}{dt}\int_V \frac{\rho}{2}\bar{u}^2 \, dV + \int_S \left(\frac{\rho}{2}\bar{u}^2 + p'\right)\bar{u}\cdot d\bar{S} = 0$$

If gravit. force

$$\partial_t \bar{u} + (\bar{u}\cdot\bar{\nabla})\bar{u} = -\frac{1}{\rho}\bar{\nabla}p + \rho\bar{g}$$

$$\bar{g} = \bar{\nabla}\chi$$

$$p' = p + \rho\chi$$

1.5 For an inviscid fluid we have Euler's equation

$$\frac{\partial \bar{u}}{\partial t} + \bar{\omega}\times\bar{u} + \bar{\nabla}\left(\frac{1}{2}\bar{u}^2\right) = -\frac{1}{\rho}\bar{\nabla}p - \bar{\nabla}\chi,$$

and, whether or not the fluid is incompressible, we also have conservation of mass:

$$\frac{D\rho}{Dt} + \rho\bar{\nabla}\cdot\bar{u} = 0.$$

Show that

$$\frac{D}{Dt}\left(\frac{\bar{\omega}}{\rho}\right) = \left(\frac{\bar{\omega}}{\rho}\cdot\bar{\nabla}\right)\bar{u} - \frac{1}{\rho}\bar{\nabla}\left(\frac{1}{\rho}\right)\times\bar{\nabla}p.$$

Deduce that, if p is a function of ρ alone, the vorticity equation is exactly as in the incompressible, constant density case, except $\bar{\omega}$ is replaced by $\bar{\omega}/\rho$.

$$\bar{\nabla}\times\left[\partial_t\bar{u} + (\bar{u}\cdot\bar{\nabla})\bar{u}\right] = \bar{\nabla}\times\left[-\frac{1}{\rho}\bar{\nabla}p - \bar{\nabla}\chi\right]$$

$$\partial_t\bar{\omega} + \bar{\nabla}\times(\bar{u}\cdot\bar{\nabla})\bar{u} = -\bar{\nabla}\times\left(\frac{1}{\rho}\bar{\nabla}p\right) - \cancel{\bar{\nabla}\times\bar{\nabla}\chi}$$

$(\bar{u}\cdot\bar{\nabla})\bar{u} = \bar{\omega}\times\bar{u} + \bar{\nabla}\left(\frac{1}{2}\bar{u}^2\right)$, where $\bar{\nabla}\times\bar{u}$

$$\bar{\nabla}\times(\bar{u}\cdot\bar{\nabla})\bar{u} = \bar{\nabla}\times(\bar{\omega}\times\bar{u}) + \cancel{\bar{\nabla}\times\bar{\nabla}\left(\frac{1}{2}\bar{u}^2\right)}$$

$$= \bar{\omega}(\bar{\nabla}\cdot\bar{u}) - \cancel{\bar{u}\underset{\bar{\nabla}\cdot\bar{\nabla}\times\bar{u}}{(\bar{\nabla}\cdot\bar{\omega})}} + (\bar{u}\cdot\bar{\nabla})\bar{\omega} - (\bar{\omega}\cdot\bar{\nabla})\bar{u}$$

$$\bar{\nabla}\times\left(\frac{1}{\rho}\bar{\nabla}p\right) = \frac{1}{\rho}\cancel{\bar{\nabla}\times\bar{\nabla}p} + (\bar{\nabla}p)\times\left(\bar{\nabla}\frac{1}{\rho}\right)$$

$$\frac{1}{\rho}\partial_t\bar{\omega} + \frac{1}{\rho}\bar{\omega}(\bar{\nabla}\cdot\bar{u}) + \frac{1}{\rho}(\bar{u}\cdot\bar{\nabla})\bar{\omega} - \frac{1}{\rho}(\bar{\omega}\cdot\bar{\nabla})\bar{u} = -\frac{1}{\rho}\left(\bar{\nabla}\frac{1}{\rho}\right)\times\bar{\nabla}p$$

$$D_t\left(\frac{\bar{\omega}}{\rho}\right) = \frac{1}{\rho}D_t\bar{\omega} - \frac{\bar{\omega}}{\rho^2}D_t\rho = \frac{1}{\rho}\left[\partial_t\bar{\omega} + (\bar{u}\cdot\bar{\nabla})\bar{\omega}\right]$$

$$+ \underbrace{\frac{\bar{\omega}}{\rho} \bar{\nabla} \cdot \bar{u}}$$

General case

$$D_t\left(\frac{\bar{\omega}}{\rho}\right) - \left(\frac{\bar{\omega}}{\rho} \cdot \bar{\nabla}\right)\bar{u} = -\frac{1}{\rho}\left(\bar{\nabla}\frac{1}{\rho}\right) \times \bar{\nabla} p$$

$$\rightarrow \underbrace{D_t \bar{\omega} - (\bar{\omega} \cdot \bar{\nabla})\bar{u} = 0}$$

Incompr. $\bar{\nabla} \cdot \bar{u} = 0$ and $\rho = \text{const}$.

If $p = p(\rho)$

$$\bar{\nabla} p = \frac{dp}{d\rho} \bar{\nabla} \rho$$

$$\bar{\nabla} \frac{1}{\rho} = -\frac{1}{\rho^2} \bar{\nabla} \rho$$

$$\left(\bar{\nabla} \frac{1}{\rho}\right) \times \bar{\nabla} p \sim \bar{\nabla} \rho \times \bar{\nabla} \rho = 0$$

$$D_t\left(\frac{\bar{\omega}}{\rho}\right) - \left(\frac{\bar{\omega}}{\rho} \cdot \bar{\nabla}\right)\bar{u} = 0$$